# Multivariate Gaussian Distribution

- Let $\mathbf{x}$ be a zero-mean random variable on $\mathbb{R}^p$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{p/2}}|R|^{-1/2}\exp\left\{-\frac{1}{2}\mathbf{x}^T R^{-1}\mathbf{x}\right\}$$

  where $R$ is the $p \times p$ covariance matrix.

- The matrix $R$ is a positive definite symmetric matrix, then

$$R = E\Lambda E^t$$

  where $E = [\mathbf{e}_1, \ldots, \mathbf{e}_p]$ is an orthonormal matrix of eigenvectors, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$ is a diagonal matrix of eigenvalues.

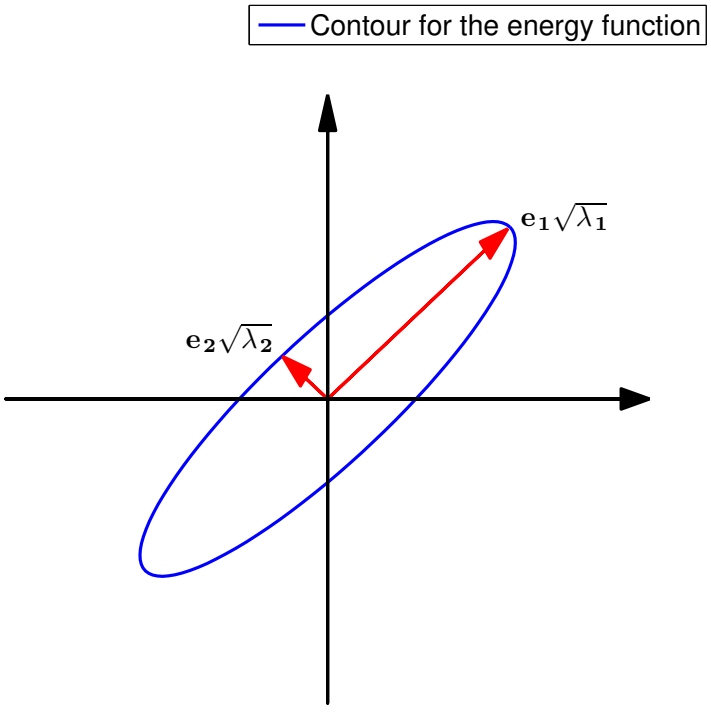
- Therefore, we have $E^t E = I$.

# Contour for the Energy Function

- Intuitively, the energy function (square of the Mahalanobis distance)

$$f(\mathbf{x}) = \mathbf{x}^t R^{-1} \mathbf{x}$$

has contour plots shown here.



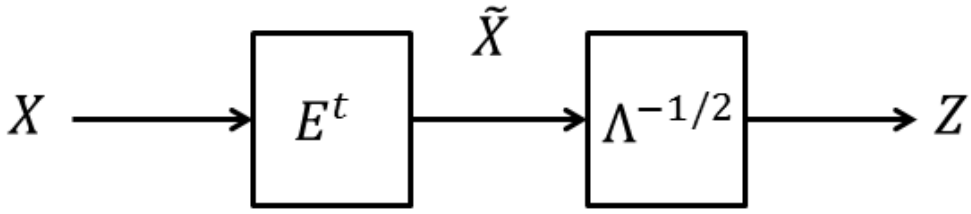
# Gaussian Random Variable Decorrelation

- Consider $\tilde{\mathbf{x}} = E^t\mathbf{x}$, then

$$\begin{aligned}\mathbb{E}[\tilde{\mathbf{x}}\ \tilde{\mathbf{x}}^t] =& \mathbb{E}[E^t\mathbf{x}\ \mathbf{x}^t E] \\ =& E^t\mathbb{E}[\mathbf{x}\ \mathbf{x}^t]E \\ =& E^t R E \\ =& E^t E \Lambda E^t E \\ =& \Lambda\end{aligned}$$

- Therefore, the elements of $\tilde{\mathbf{x}}$ are uncorrelated with variance $\mathbb{E}[\tilde{x}_i^2] = \Lambda_{ii}$.
- The elements of $\tilde{\mathbf{x}}$ are independent, since $\tilde{\mathbf{x}}$, as the linear transform of $\mathbf{x}$, is Gaussian distributed.

# Whitening Gaussian Random Variables




$$\mathbb{E}[Z\ Z^t] = I$$

- So $E^t$ decorrelates $\mathbf{x}$, while $\Lambda^{-\frac{1}{2}}E^t$ whitens $\mathbf{x}$.

$$E^t\mathbf{x} = \begin{bmatrix} \mathbf{e}_1^t \\ \vdots \\ \mathbf{e}_p^t \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \\ \end{bmatrix}$$

- The eigenvectors $\mathbf{e}_k$, called eigen-signals, are basis vectors to represent the signal $\mathbf{x}$.
- If $\mathbf{x}$ represents an image, then the eigenvectors $\mathbf{e}_k$ are also called *eigenimages*.

# Eigenimage Estimation

- Assume we have $n$ training vectors $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$, then

$$\begin{aligned} R_x =& E[\mathbf{x}_k \, \mathbf{x}_k^t] \\ =& E[XX^t]/n \\ \cong& XX^t/n \\ =& S \end{aligned}$$

  where $S = \frac{1}{n} XX^t$ is the sample correlation matrix

$$S_{ij} = \frac{1}{n} \sum_{k=1}^{n} X_{ik} X_{jk}$$

- Decompose $S$ as

$$S = \hat{E} \hat{\Lambda} \hat{E}^t$$

  where $\hat{E}$ is an estimate of the eigenvectors, and $\hat{\Lambda}$ is an estimate of the eigenvalues.

- $E$ could be very large, especially when $X$ represents $n$ images.

# Singular Value Decomposition (SVD)

- For $n < p$ it looks like

$$\underset{p \times n}{\begin{bmatrix} X \end{bmatrix}} = \underset{p \times n}{\begin{bmatrix} U \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} \Sigma \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} V^t \end{bmatrix}}$$

  - The columns of $U$ are orthonormal and called left hand singular vectors.
  - The columns of $V$ are orthonormal and called right hand singular vectors.
  - $\Sigma$ is diagonal matrix of singular values.

# Eigenimage Estimation using SVD

- Notice that

$$\begin{aligned} XX^t =& U\Sigma V^t V\Sigma U^t \\ =& U\Sigma^2 U^t \end{aligned}$$

  So $U$ is the set of the desired eigenvectors of $XX^t$.

- How to compute $U$?
  - Notice that $X^tX = V\Sigma^2V^t$ is a $n \times n$ matrix, and $V$ contains eigenvectors of a much smaller matrix.
  - Algorithm
    - Find eigenvectors $V$ of the matrix $X^tX$
    - Compute $XV = U\Sigma$, Then the columns of $U$ are the normalized columns of $XV$, and $\Sigma$ are the normalization factors.