PURDUE

ECE 63700

Final Exam, May 4, Spring 2026

NAME _____

PUID _____

Exam instructions:

- A fact sheet is included at the end of this exam for your use.
- You have 120 minutes to work the exam.
- This is a closed-book and closed-note exam. You may not use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You may not communicate with any person other than the official proctor during the exam.

To ensure Gradescope can read your exam:

- Write your full name and PUID above and on the top of every page.
- Answer all questions in the area designated for each problem.
- Write only on the front of the exam pages.
- DO NOT run over to the next question.

Name/PUID: Key

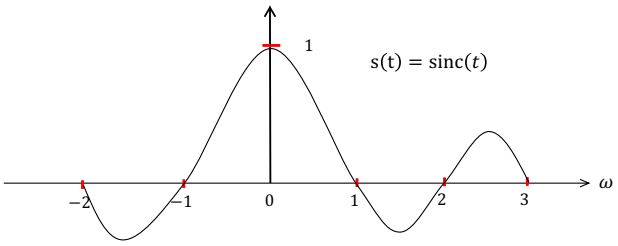
Problem 1. (40pt) Sampling

Consider the continuous time signal $s(t) = \text{sinc}(t)$ that is sampled with a period of T to yield $x(n) = s(nT)$ and time is measured in units of seconds (sec).

Note: All answers should have valid units when appropriate.

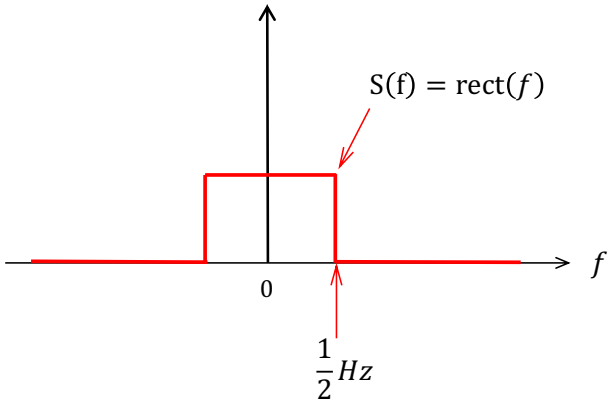
Problem 1a) Sketch a plot of the function $s(t)$.

Solution:



Problem 1b) Calculate the CTFT $S(f)$ and sketch a plot of this function.

Solution: $S(f) = \text{rect}(f)$



Problem 1c) What is the sampling frequency?

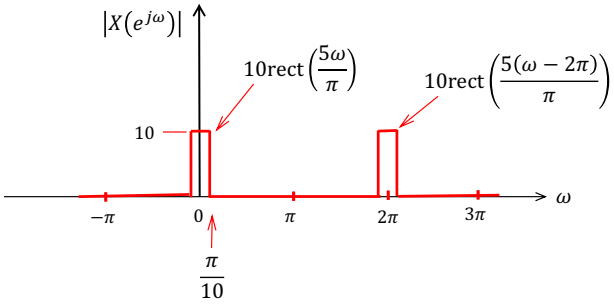
Solution: The sampling frequency is $f_s = 1/T Hz$.

Problem 1d) What is the Nyquist sampling frequency for this signal? What is the Nyquist sampling period for this signal?

Solution: $f_s = 1Hz$ and $T = 1sec.$

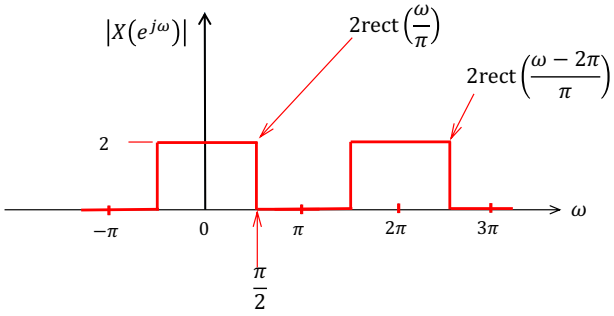
Problem 1e) Sketch $x(n)$ and $|X(e^{j\omega})|$ when $T = 0.1sec.$

Solution:

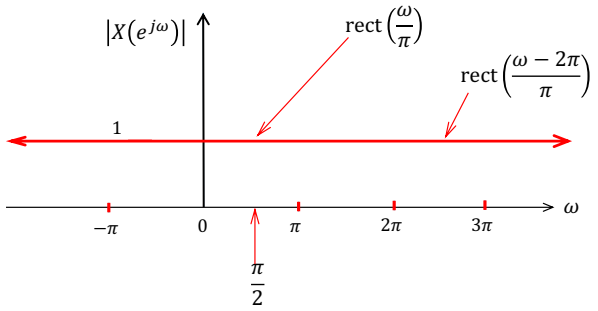


Problem 1f) Sketch $x(n)$ and $|X(e^{j\omega})|$ when $T = 0.5sec.$

Solution:



Problem 1g) Sketch $x(n)$ and $|X(e^{j\omega})|$ when $T = 1.0sec.$


$|X(e^{j\omega})|$
rect $\left(\frac{\omega}{\pi}\right)$
rect $\left(\frac{\omega - 2\pi}{\pi}\right)$
1
ω
$-\pi$
0
π
2π
3π
$\frac{\pi}{2}$

Solution:

Problem 2. (40pt) MMSE Prediction and LS Estimation

Consider an algorithm for deblurring an image Y to recover an image X where Y is assumed to be random.

For each pixel in $s \in S$, we form a vector z_s that is the pixels in a $(2p+1) \times (2p+1)$ window about the pixel s. More specifically,

$$z_s = [Y_{s+r_1}, \ldots, Y_{s+r_P}] \ ,$$

where $P = (2p+1)^2$ and $r_1, \ldots, r_P$ are the offsets associate with each pixel in the window.

Furthermore, define

$$X = \begin{bmatrix} X_{s_1} \\ \vdots \\ X_{s_N} \end{bmatrix} \quad Z = \begin{bmatrix} z_{s_1} \\ \vdots \\ z_{s_N} \end{bmatrix}$$

where $s_1, \ldots, s_N$ index the pixels in the image.

Then are deblurred image is formed by $\hat{X} = Z\theta$.

Problem 2a) Is $\hat{X}$ random or deterministic? Why?

Solution: It is random because Z is random, so it is a function of a random quantity.

Problem 2b) Is θ normally assumed to be random or deterministic?

Solution: θ is normally assumed to be deterministic but unknown.

Problem 2c) Calculate a closed form expression (that I like) for the Mean Squared Error,

$$MSE = \frac{1}{N} E[\|X - \hat{X}\|^2] \ .$$

Solution:

$$\begin{aligned} MSE &= \frac{1}{N} E\left[\|X - Z\theta\|^2\right] \\ &= \frac{1}{N} E\left[(X - Z\theta)^t (X - Z\theta)\right] \\ &= \frac{1}{N} E\left[X^t X - 2X^t Z\theta + \theta^t Z^t Z\theta\right] \\ &= a - 2b^t\theta + \theta^t R\theta \end{aligned}$$

where

$$a = \frac{1}{N} E\left[\|X\|^2\right]$$
$$b = \frac{1}{N} E\left[X^t Z\right]$$
$$R = \frac{1}{N} E\left[Z^t Z\right] \ .$$

Problem 2d) Calculate an expression for the value θ^* that minimizes the MSE.

Solution:

$$\nabla_\theta MSE = -2b + 2R\theta^* = 0$$

which yields

$$\theta^* = R^{-1} b \ .$$

Problem 2e) Is θ^* from the previous subproblem random or deterministic? Why?

Solution: θ^* is deterministic because the expectation is always deterministic so b and R are not random.

Problem 2f) Calculate a closed form expression (that I like) for the Average Squared Error,

$$ASE = \frac{1}{N} \sum_{n=1}^{N} (X_{s_n} - \hat{X}_{s_n})^2 \ .$$

Solution:

$$\begin{aligned} ASE &= \frac{1}{N} \|X - Z\theta\|^2 \\ &= \frac{1}{N} (X - Z\theta)^t (X - Z\theta) \\ &= \frac{1}{N} X^t X - 2X^t Z\theta + \theta^t Z^t Z\theta \\ &= \hat{a} - 2\hat{b}^t \theta + \theta^t \hat{R} \theta \end{aligned}$$

where

$$\hat{a} = \frac{1}{N}\|X\|^2$$
$$\hat{b} = \frac{1}{N}X^t Z$$
$$\hat{R} = \frac{1}{N}Z^t Z \ .$$

Problem 2g) Calculate an expression for the value $\hat{\theta}$ that minimizes the ASE.

Solution:

$$\nabla_\theta MSE = -2b + 2R\hat{\theta} = 0$$

which yields

$$\hat{\theta} = \hat{R}^{-1}\hat{b} \ .$$

Problem 2h) Is $\hat{\theta}$ from the previous subproblem random or deterministic? Why?

Solution: $\hat{\theta}$ is random because it is a function of random quantities $\hat{b}$ and $\hat{R}$.

Problem 3. (25pt) MR Imaging

Consider an MRI that only images in one dimension, x. So for example, the object being imaged might be a thin rod oriented along the x-dimension that extends from $-\infty$ to ∞.[1] In this example, assume that the magnetic field strength at each location is given by

$$M(x,t) = M_o + xG(t)$$

where M_o is the static magnetic field strength and $G(t)x$ is the linear gradient field in the x dimension. Furthermore, let:

γ denote the gyromagnetic constant for hydrogen;

$a(x)$ denote the quantity of hydrogen per unit distance along the length of the rod;

$\omega_o = \gamma M_o$ be the center frequency of the scanner in units of rad/sec.

$k(t) = \gamma \int_0^t G(\tau)d\tau$ is the position in k-space at time t.

$A(f) = \text{CTFT}\{a(t)\}$ is the CTFT of $a(t)$.

Problem 3a) Calculate a simplified expression for $\omega(x,t)$, the precession frequency of a hydrogen atom at location x and time t.

Solution:

$$\begin{aligned}\omega(x,t) &= \gamma M(x,t) \\ &= \gamma\left[M_o + xG(t)\right] \\ &= \omega_o + x\gamma G(t)\end{aligned}$$

Problem 3b) Calculate $\phi(x,t)$, the phase of precession of a hydrogen atom at location x and time t assuming that $\phi(x,0) = 0$.

Solution:

$$\begin{aligned}\phi(x,t) &= \int_0^t \omega(x,\tau)d\tau \\ &= \int_0^t \left[\omega_o + \gamma\, G(t)\right] d\tau \\ &= \omega_o t + x\,\gamma \int_0^t G(\tau)d\tau \\ &= \omega_o t + x\, k(t)\end{aligned}$$

[1] This is not physically possible, but don't worry about that. Just do the math that way for simplicity.

Problem 3c) Calculate $r(x,t)$, the signal radiated from hydrogen atoms in the interval $[x, x+dx]$ at time t.

Solution:

$$r(x,t) = a(x)e^{j\phi(x,t)}$$

Problem 3d) Calculate $r(t)$, the total signal radiated from hydrogen atoms along the entire object.

Solution:

$$\begin{aligned} r(t) &= \int_{-\infty}^{\infty} a(x)e^{j\phi(x,t)}dx \\ &= \int_{-\infty}^{\infty} a(x)e^{j\omega_o t + jx\,k(t)}dx \\ &= e^{j\omega_o t}\int_{-\infty}^{\infty} a(x)e^{jx\,k(t)}dx \\ &= e^{j\omega_o t}\int_{-\infty}^{\infty} a(x)e^{-j2\pi x\,\frac{-k(t)}{2\pi}}dx \\ &= e^{j\omega_o t}A(-k(t)/(2\pi)) \end{aligned}$$

Problem 3e) Describe in words and equations how to reconstruct the signal $a(t)$ from the signal $r(t)$.

Solution: First demodulated the signal to form $s(t) = r(t)e^{-\omega_o t}$. Then from $s(t)$ compute $A(f)$ so that

$$A(f)|_{f=-k(t)/(2\pi)} = s(t)\ .$$

Then take the inverse DTFT of $A(f)$ to form $a(t)$.

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Problem 4. (45pt) Linear Systems

Consider the following two discrete time systems T_1 and T_2 where

$$y = T_1[x] \text{ is defined by } : \; y_n = \sum_{k=-p}^{p} a_k x_{n-k} \tag{1}$$

$$y = T_2[x] \text{ is defined by } : \; y_n = \sum_{k=-p}^{p} b_k x_{n-k} \tag{2}$$

Then define the new systems T_3 and T_4 by

$$T_3 = T_1 + T_2 \ ,$$

and

$$T_4 = T_1 \circ T_2 \ ,$$

where $\circ$ denotes the composition of functions.

Problem 4a) Prove that T_1 is a linear system.

Solution: Let x_n^1, x_n^2, be any DT signals and let α, β be any real valued numbers. Then we have that

$$\begin{aligned} T_1[\alpha x^1 + \beta x^2] &= \sum_{k=-p}^{p} a_k(\alpha x_{n-k}^1 + \beta x_{n-k}^2) \\ &= \alpha \sum_{k=-p}^{p} a_k x_{n-k}^1 + \beta \sum_{k=-p}^{p} a_k x_{n-k}^2 \\ &= \alpha T_1[x^1] + \beta T_1[x^2] \end{aligned}$$

Problem 4b) Prove that T_1 is a time invariant system.

Solution: Let $x_n^d = s_{n-d}$ and let $y^d = T_1[x^d]$. Then we need to show that

$$y_n^d = y_{n-d}^0$$

So then

$$\begin{aligned} y_n^d &= T_1[x^d] \\ &= T_1[s_{n-d}] \\ &= \sum_{k=-p}^{p} a_k \alpha s_{n-k-d} \\ &= y_{n-d}^0 \ . \end{aligned}$$

Problem 4c) What is the DC gain of the system T_1?

Solution:

$$\text{DC gain} = A(e^{j0}) = \sum_{k=-\infty}^{\infty} a_k$$

Problem 4d) What DC gain does one typically choose for a LTI system used to process an image? Why?

Solution: The DC gain is typically taken to be 1 because typically you do not want to change the brightness of the image.

Problem 4e) What is the impulse response of T_1?

Solution: The impulse response is given by a_k.

Problem 4f) Is T_3 and LTI system?

Solution: Yes, T_3 is LTI because the sum of two LTI systems is LTI.

Problem 4g) What is the impulse response of T_3? (Denote the impulse response by h_n.)

Solution: The impulse response of T_3 is $h_n = a_n + b_n$.

Problem 4h) Is T_4 and LTI system?

Solution: Yes, T_4 is LTI because the composition of two LTI systems is LTI.

Problem 4i) What is the impulse response of T_4? (Denote the impulse response by h_n.)

Solution: The impulse response of T_3 is

$$h_n = \sum_{k=-\infty}^{\infty} h_k g_{n-k}$$

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X^*(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi ft_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

$$(n+1)a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1-ae^{-j\omega})^2}$$

- Rep and Comb relations

$$\text{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\text{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\text{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}}[X(f)]$$

$$\text{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(f) &= Y\left(e^{j2\pi fT}\right) \end{aligned}$$