# PURDUE

ECE 63700

Exam #2, April 10, Spring 2026

NAME _____

PUID _____

**Exam instructions:**

- A fact sheet is included at the end of this exam for your use.
- You have 50 minutes to work the exam.
- This is a closed-book and closed-note exam. You may not use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You may not communicate with any person other than the official proctor during the exam.

**To ensure Gradescope can read your exam:**

- Write your full name and PUID above and on the top of every page.
- Answer all questions in the area designated for each problem.
- Write only on the front of the exam pages.
- DO NOT run over to the next question.

**Name/PUID:** **Key**

**Problem 1. (40pt) White Noise Driven Random Process**

Let $Y_n$, $X_n$, and $W_n$ be discrete time random processes. Furthermore, let $W_n$ have i.i.d. samples with distribution $N(0,1)$, and let $Y = T[W]$ where $y = T[w]$ is a causal system defined by

$$y_n = w_n + \sum_{k=1}^{p} h_k y_{n-k}$$

for some set of filter coefficients $h_k$ for $k = 1, \ldots, p$. Also, define $Y_{i<n} = \{Y_{n-1}, Y_{n-2}, \ldots\}$.

**Problem 1a)** Is $T$ a linear time invariant system? Justify your answer.

---

**Solution:**
Yes, $T$ is both linear and time invariant.

---

**Problem 1b)** Calculate $H(e^{j\omega})$ the frequency response of $T$.

---

**Solution:**

$$\begin{aligned} y(z) &= w(z) + \sum_{k=1}^{p} h_k z^{-k} y(z) \\ y(z) - \sum_{k=1}^{p} h_k z^{-k} y(z) &= w(z) \\ y(z)\left(1 - \sum_{k=1}^{p} h_k z^{-k}\right) &= w(z) \\ H(z) = \frac{y(z)}{x(z)} &= \frac{1}{1 - \sum_{k=1}^{p} h_k z^{-k}} \end{aligned}$$

So we have that

$$H(e^{j\omega}) = \frac{1}{1 - \sum_{k=1}^{p} h_k e^{-jk\omega}}$$

---

**Problem 1c)** Calculate $S_W(e^{j\omega})$ the power spectrum of $W_n$.

---

**Solution:**
$S_W(e^{j\omega}) = 1$

---

**Problem 1d)** Calculate $S_Y(e^{j\omega})$ the power spectrum of $Y_n$.

---

**Solution:**

$$\begin{aligned} S_Y(e^{j\omega}) &= S_W(e^{j\omega}) \left|H(e^{j\omega})\right|^2 \\ &= 1 \left| \frac{1}{1 - \sum_{k=1}^{p} h_k e^{-jk\omega}} \right|^2 \\ &= \frac{1}{\left|1 - \sum_{k=1}^{p} h_k e^{-jk\omega}\right|^2} \end{aligned}$$

---

**Problem 1e)** Calculate the value of $E\left[W_n | Y_{i<n}\right]$.

---

**Solution:**
Since the system $T$ is causal, $W_n$ is independent of $Y_{i<n}$, so we have that

$$E\left[W_n | Y_{i<n}\right] = E\left[W_n\right] = 0$$

---

**Problem 1f)** Calculate an expression for

$$E\left[Y_n | Y_{i<n}\right] \ .$$

---

**Solution:**

$$
\begin{aligned}
E\left[Y_n|Y_{i<n}\right] &= E\left[W_n + \sum_{k=1}^{p} h_k Y_{n-k}|Y_{i<n}\right] \\
&= E\left[W_n|Y_{i<n}\right] + E\left[\sum_{k=1}^{p} h_k Y_{n-k}|Y_{i<n}\right] \\
&= 0 + \sum_{k=1}^{p} h_k E\left[Y_{n-k}|Y_{i<n}\right] \\
&= \sum_{k=1}^{p} h_k Y_{n-k}
\end{aligned}
$$

---

**Problem 1g)** Calculate an expression for

$$Y_n - E\left[Y_n|Y_{i<n}\right] \ .$$

---

**Solution:**

$$
\begin{aligned}
Y_n - E\left[Y_n|Y_{i<n}\right] &= Y_n - \sum_{k=1}^{p} h_k Y_{n-k} \\
&= W_n
\end{aligned}
$$

---

**Problem 1h)** What does problem 1g) above tell you about the prediction errors for the random process $Y_n$?

---

**Solution:**
It says that the prediction errors $W_n$ are independent and identically distributed with zero mean and unit variance.

---

**Name/PUID:** _____

**Problem 2. (30pt) Multivariate Gaussian**

Let $X \sim N(0, R)$ be a 2−dimensional Gaussian random vector with covariance $R$. Then define

$$Y = T_1 X$$
$$Z = T_2 Y$$
$$W = T_3 Z$$

where $Y = T_1 X$ where $T_1 = \text{diag}\{1/\sqrt{R_{0,0}}, 1/\sqrt{R_{1,1}}\}$. (Note: All indexing starts at 0.)

**Problem 2a)** What are the variance of the components $Y_0$ and $Y_1$?

---

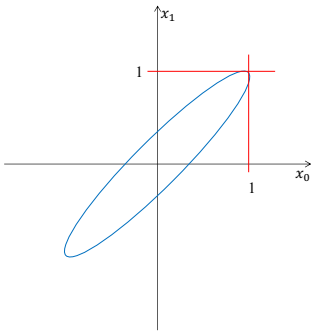
**Solution:**
$E[Y_0^2] = E[Y_1^2] = 1$

---

**Problem 2b)** Sketch the 2D distribution of $Y$ with a contour plot.

---

**Solution:**




---

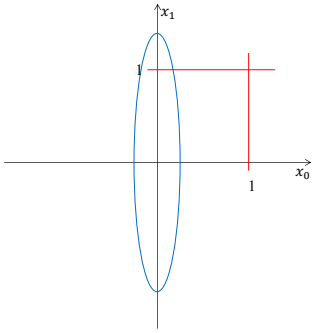
**Problem 2c)** Find an explicit 2D linear orthonormal transformation for $T_2$ so that the components of $Z$ are independent.

**Solution:**

$$T_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

**Problem 2d)** Sketch the 2D distribution of $Z$ with a contour plot.

**Solution:**



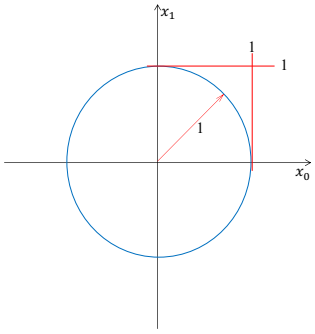

**Problem 2e)** Find a 2D linear transformation for $T_3$ so that the components of $W$ are independent and identically distributed with variance 1.

**Solution:**

$$T_0 = \begin{bmatrix} \frac{1}{\sqrt{E[Z_0^2]}} & 0 \\ 0 & \frac{1}{\sqrt{E[Z_1^2]}} \end{bmatrix}$$

**Problem 2f)** Sketch the 2D distribution of $W$ with a contour plot.

**Solution:**




---

**Name/PUID:** _____
**Problem 3. (20pt) Colormetric Transformations**

Let $I \in \Re^{31}$ represent the spectrum of light sample in 10nm steps from 400nm to 700nm. Our goal will be to match the color of this spectrum of light using three SMPTE 709 RGB color primaries, $\mathbf{R}, \mathbf{G}, \mathbf{B}$. We will also use the $X, Y, Z$ color system to do this.

Furthermore, assume that

$$\begin{bmatrix} \mathrm{r} \\ \mathrm{g} \\ \mathrm{b} \end{bmatrix} = T_1 I$$

$$\begin{bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{bmatrix} = T_2 I$$

**Problem 3a)** What name do we give to the rows of $T_1$?
Also, list out two properties for the rows of $T_1$.

---

**Solution:**

The rows of $T_1$ are called the r,g,b, color matching functions, and they have the following two properties:

- sum to 1
- are both positive and negative

---

**Problem 3b)** What name do we give to the rows of $T_2$?
Also, list out two properties for the rows of $T_2$.

---

**Solution:**

The rows of $T_2$ are called the X,Y,Z color matching functions, and they have the following two properties:

- sum to 1
- are non-negative

---

**Problem 3c)** Which transformation $T_1$ or $T_2$ is practical to implement in a camera? Why?

---

**Solution:**
$T_2$ is practical to implement in a camera because the color matching functions are non-negative.

---

**Problem 3d)** You are designing a color display. Which color space can be directly implemented, $X, Y, Z$ or $r, g, b$? Why?

---

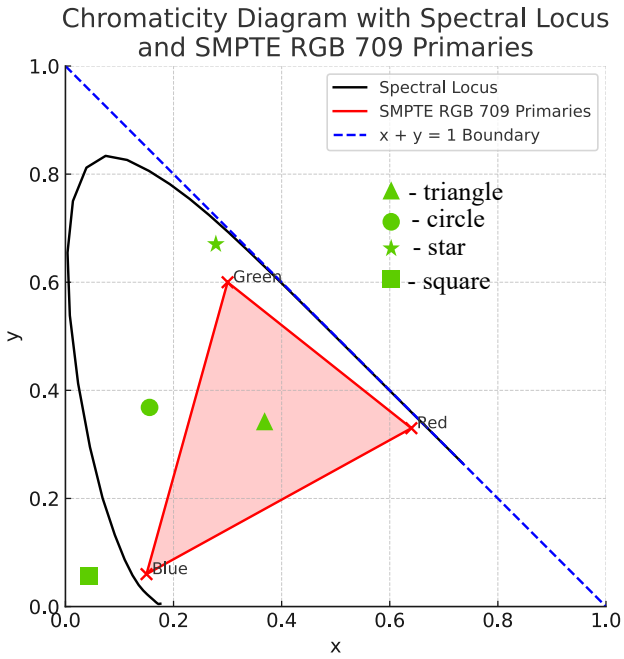
**Solution:**
The $r, g, b$ color space is practical for implementation on a display because the color primaries are real. However, $X, Y, Z$ color space is not practical for implementation on a display because the color primaries are imaginary.

---

**Name/PUID:** _____

**Problem 4. (20pt) Color Vision**

In this problem, assume you are using an emissive display with SMPTE 709 RGB color primaries. Also, refer to the following chromaticity diagram.




**Problem 4a)** What color is represented by the triangle?

_____

**Solution:**
White

_____

**Problem 4b)** Let $(r_c, g_c, b_c)$ denote the tristimuluous values indicated by the **circle**.

- Is this a real color?
- Can this color be displayed?
- Are the values of $r_c$, $g_c$, and $b_c$ less than or greater than zero?

_____

**Solution:**
Yes, the color is real. No, the color can not be displayed. $r_t < 0$, $g_t > 0$, and $b_t > 0$

---

**Problem 4c)** Let $(r_s, g_s, b_s)$ denote the tristimuluous values indicated by the **star**.

- Is this a real color?
- Can this color be displayed?
- Are the values of $r_s$, $g_s$, and $b_s$ less than or greater than zero?

---

**Solution:**
Yes, the color is real. No, the color can not be displayed. $r_t < 0$, $g_t > 0$, and $b_t < 0$

---

**Problem 4d)** Let $(r_q, g_q, b_q)$ denote the tristimuluous values indicated by the **square**.

- Is this a real color?
- Can this color be displayed?
- Are the values of $r_q$, $g_q$, and $b_q$ less than or greater than zero?

---

**Solution:**
No, the color is not real. No, the color can not be displayed. $r_t < 0$, $g_t > 0$, and $b_t > 0$

---

# Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X^*(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi ft_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

$$(n+1)a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1-ae^{-j\omega})^2}$$

- Rep and Comb relations

$$\text{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\text{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\text{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}}[X(f)]$$

$$\text{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(f) &= Y\left(e^{j2\pi fT}\right) \end{aligned}$$