EE 438 Final Exam #3
December 12, Fall 2007

Name: Solutions

Instructions:

- Follow all instructions carefully!
- This is a 110 minute exam containing **six** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Function types
  - *Right-sided:* $\exists N s.t. \forall n < N, x(n) = 0$
  - *Left-sided:* $\exists N s.t. \forall n > N, x(n) = 0$
  - *Causal:* $\forall n < 0, x(n) = 0$
  - *Anticausal:* $\forall n > 0, x(n) = 0$
  - *Two-sided:* {x(n) is not right-sided} and {x(n) is not left-sided}

- CTFS

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

- CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_o t} \overset{CTFT}{\Leftrightarrow} X(f-f_o)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

$$e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{j2\pi f + a}$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- DTFT Properties

$$x(-n) \overset{DTFT}{\Leftrightarrow} X(e^{-j\omega})$$

$$x(n-n_o) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})e^{-j\omega n_0}$$

$$x(n)e^{j\omega_o t} \overset{DTFT}{\Leftrightarrow} X(e^{j(\omega-\omega_o)})$$

$$x(n)y(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{2\pi}X(e^{j\omega}) * Y(e^{j\omega})$$

$$x(n) * y(n) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})Y(e^{j\omega})$$

- Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

- DFT

$$X_k = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right)$$

$$s(t) = \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT)$$

$$S(f) = Y\left(e^{j2\pi fT}\right)$$

- Decimation and Interpolation

$$
\begin{aligned}
y(n) &= x(nD) \\
Y(e^{j\omega}) &= \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j\frac{\omega-2\pi k}{D}}\right) \\
s(n) &= \sum_{k=-\infty}^{\infty} y(k)\delta(n-kL) \\
S(e^{j\omega}) &= Y(e^{j\omega L})
\end{aligned}
$$

Name: Solutions

Problem 1.(20pt) For each of the following discrete-time systems with input $x(n)$ and output $y(n)$, state whether the system is i) linear or non-linear, ii) time-invariant or time varying, iii) memoryless or memory, and iv) BIBO stable or unstable. (No derivation or proof is required for this problem.)

a) $y(n) = x(n)$

b) $y(n) = \frac{1}{x(n)}$

c) $y(n) = \sum_{k=-\infty}^{n} x(k)$

d) $y(n) = x(n)e^{-n}$

e) $y(n) = e^{x(n)}$.

a) $y(n) = x(n)$
i) linear
ii) time-invariant
iii) memoryless
iv) BIBO stable

b) $y(n) = \frac{1}{x(n)}$
i) non-linear
ii) time-invariant
iii) memoryless
iv) unstable

c) $y(n) = \sum_{k=-\infty}^{n} x(k) = \sum_{k=-\infty}^{\infty} x(k)(1)^n u(n-k) \Rightarrow h(n) = 1^n u(n)$
i) linear
ii) time-invariant
iii) memory
iv) unstable

d) $y(n) = x(n)e^{-n}$
i) linear
ii) time varying
iii) memoryless
iv) unstable

e) $y(n) = e^{x(n)}$
i) non-linear
ii) time-invariant
iii) memoryless
iv) BIBO stable

Name: Solutions

Problem 2.(10pt)
Let $y(n) = T[x(n)]$ be a discrete time LTI system with input $x(n)$ and output $y(n)$, and let $h(n) = T[\delta(n)]$ be the response to an impulse at time zero. Derive an expression for the output $y(n)$ for any input $x(n)$.

$$y(n) = T[x(n)]$$

$$= T[\ldots + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + \ldots]$$

$$= T\left[\sum_{K=-\infty}^{\infty} x(K)\delta(n-K)\right]$$

$$= \sum_{K=-\infty}^{\infty} x(K)\, T[\delta(n-K)]$$ because the system is linear

$$= \sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$$ because the system is time-invariant

$$\therefore y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K) = x(n) * h(n)$$

Name: Solutions

Problem 3.(10pt)

Let $y(n) = x(n) * h(n)$ be a discrete-time LTI system with input $x(n)$ and output $y(n)$.

a) Prove that when the input has the form $x(n) = e^{j\omega_o n}$ the output has the form $y(n) = Ce^{j\omega_o n}$ and derive an expression for C.

b) Use this result to calculate $y(n)$ when $x(n) = \cos(\omega_o n)$ where $C = Ae^{j\theta}$.

a) $y(n) = x(n) * h(n)$ → By definition of DT convolution and associate property

$$y(n) = \sum_{K=-\infty}^{\infty} x(n-K)\,h(K)$$

→ Substituting in for $x(n-K)$

$$y(n) = \sum_{K=-\infty}^{\infty} e^{j\omega_o(n-K)}\,h(K)$$

→ By commutative property of multiplication

$$y(n) = e^{j\omega_o n} \sum_{K=-\infty}^{\infty} e^{-j\omega_o K}\,h(K)$$

→ By definition of the DTFT

$$y(n) = e^{j\omega_o n}\,H(e^{j\omega_o})$$

where $H(e^{j\omega}) = \text{DTFT}\{h(n)\}$

$$\Rightarrow \boxed{C = H(e^{j\omega_o})}$$

b) $x(n) = \cos(\omega_o n) = \frac{1}{2}e^{j\omega_o n} + \frac{1}{2}e^{-j\omega_o n}$

$$\Rightarrow y(n) = \frac{1}{2}H(e^{j\omega_o})e^{j\omega_o n} + \frac{1}{2}H(e^{-j\omega_o})e^{-j\omega_o n}$$

$$= \frac{1}{2}|H(e^{j\omega_o})|\,e^{j\angle H(e^{j\omega_o})}\,e^{j\omega_o n} + \frac{1}{2}|H(e^{-j\omega_o})|\,e^{j\angle H(e^{-j\omega_o})}\,e^{-j\omega_o n}$$

$$\boxed{y(n) = \frac{1}{2}|H(e^{j\omega_o})|\,e^{j(\angle H(e^{j\omega_o})+\omega_o n)} + \frac{1}{2}|H(e^{-j\omega_o})|\,e^{j(\angle H(e^{-j\omega_o})-\omega_o n)}}$$

If $h(n)$ is real, then $H^*(\omega) = H(-\omega)$

$$\Rightarrow y(n) = \frac{1}{2}|H(e^{j\omega_o})|\left(e^{j(\angle H(e^{j\omega_o})+\omega_o n)} + e^{-j(\angle H(e^{j\omega_o})+\omega_o n)}\right)$$

$$y(n) = |H(e^{j\omega_o})|\cos(\omega_o n + \angle H(e^{j\omega_o}))$$

Solutions

Problem 4.(20pt)
Consider the discrete-time LTI system described by the difference equation

$$y(n) = ay(n-1) + x(n)$$

where a is a real value, $x(n)$ is the input and $y(n)$ is the output.

a) Find the transfer function $H(z)$ of the system.

b) Compute the right sided impulse response of the system and specify its ROC.

c) Compute the left sided impulse response of the system and specify its ROC.

d) In this part, we will assume that the system is known to be stable. Give the impulse response for $a = 2$.

a) $y(n) = a\,y(n-1) + x(n)$

$\Rightarrow Y(z) = a z^{-1} Y(z) + X(z)$

$Y(z)(1 - a z^{-1}) = X(z)$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}}}$$

b) If the system is right-sided the ROC must be $\boxed{\Omega_H = \{z \in \mathbb{C} \mid |z| > |a|\}}$

$$H(z) = \frac{1}{1 - a z^{-1}} = \sum_{n=0}^{\infty} (a z^{-1})^n = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=-\infty}^{\infty} (a^n u(n)) z^{-n}$$

$$\Rightarrow \boxed{h(n) = a^n u(n)}$$

c) If the system is left-sided, the ROC must be

$\Omega_H = \{z \in \mathbb{C} \mid |z| < |a|\}$.

$$H(z) = \frac{1}{1 - a z^{-1}} = \frac{\frac{z}{a}}{\frac{z}{a} - 1} = 1 - \frac{1}{1 - \frac{z}{a}} = 1 - \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n = 1 - \sum_{n=-\infty}^{0} a^n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(-a^n u(-n-1)\right) z^{-n} \xrightarrow{10} \boxed{h(n) = -a^n u(-n-1)}$$

Name: Solutions

d) If the system is stable, then $|z|=1 \subset \Omega_H$ and since $|a|>1$, the ROC must be $|z|<a$.
$\Rightarrow$ the system is left-sided, and has the form.

$h(n) = -a^n u(-n-1)$

$\boxed{h(n) = -2^n u(-n-1)}$

Problem 5.(20pt)
Let $x(n) = u(n) - u(n-L)$.

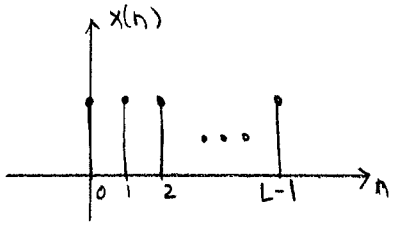
a) Calculate $X(z)$, the z-transform of $x(n)$, and specify the ROC.

b) Calculate $X(e^{j\omega})$, the DTFT of $x(n)$.

c) Sketch $|X(e^{j\omega})|$ for $L = 5$ on the interval $[0, 2\pi]$.

d) Calculate $X_N(k)$ the DFT of $x(n)$ for $N \geq L$.

e) Sketch $|X_N(k)|$ the magnitude of the DFT of $x(n)$ for $N = 10$. and $L = 5$



(a) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

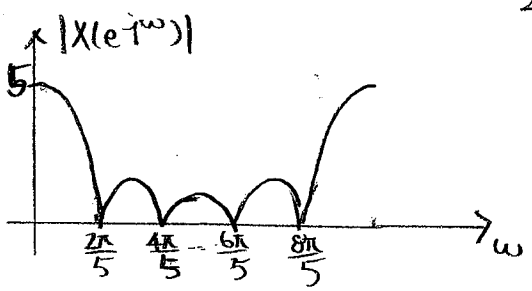
$= \sum_{n=0}^{L-1} z^{-n} = \sum_{n=0}^{L-1} \left(\frac{1}{z}\right)^n = \frac{1 - \left(\frac{1}{z}\right)^{L-1}\left(\frac{1}{z}\right)}{1 - \frac{1}{z}} = \frac{1 - \left(\frac{1}{z}\right)^{L}}{1 - \frac{1}{z}}$

$= \frac{1 - z^{-L}}{1 - z^{-1}}$; ROC: $\{z \in \mathbb{C} \mid |z| > 0\}$

(b) $X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}} = \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} = \frac{e^{-j\frac{\omega L}{2}}\left(e^{+j\frac{\omega L}{2}} - e^{-j\frac{\omega L}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$

$= e^{-j\left(\frac{\omega L}{2} - \frac{\omega}{2}\right)} \frac{\sin\left(\frac{\omega L}{2}\right)}{\sin\frac{\omega}{2}} = e^{-j\omega\left(\frac{L-1}{2}\right)} \frac{\sin\left(\frac{\omega L}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$

(c)



$|X(e^{j\omega})| = \left|\frac{\sin\left(\frac{\omega L}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right|$

Zeros at $\frac{\omega L}{2} = m\pi$ $m = 1, 2, \ldots$

$\omega = m\frac{2\pi}{L}$

$= m\frac{2\pi}{5}$

Name: Solutions

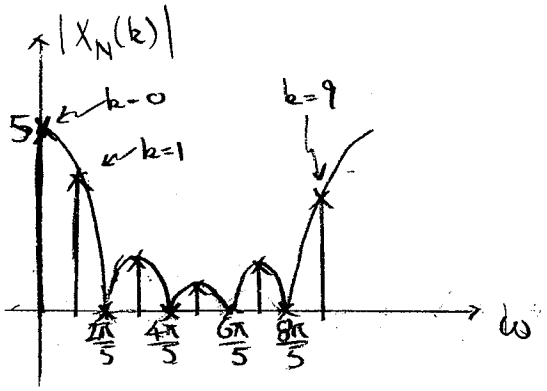
(d) $X_N(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$

$= e^{-j\frac{2\pi k}{N}\left(\frac{L-1}{2}\right)} \dfrac{\sin\left(\frac{2\pi k}{N}\frac{L}{2}\right)}{\sin\left(\frac{2\pi k}{2N}\right)} = e^{-j\frac{2\pi k}{N}\left(\frac{L-1}{2}\right)} \dfrac{\sin\left(\frac{L\pi k}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}$

(e) $|X_N(k)| = \left|\dfrac{\sin\left(\frac{5\pi k}{10}\right)}{\sin\left(\frac{2\pi k}{10}\right)}\right|$

$= \left|\dfrac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{10}\right)}\right|$

$k = 0, 1, \ldots, 9$



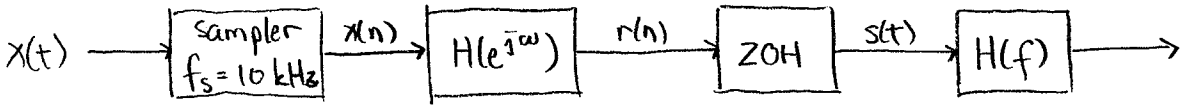
Solutions

Problem 6.(20pt)
A CT signal $x(t)$ is sampled at 10 kHz to form a DT signal $y(n)$. The signal $y(n)$ is then filtered with the DT filter $H(e^{j\omega})$ to form a signal $r(n)$. Finally, the signal $r(n)$ is reconstructed to form $s(t)$ by using a ZOH followed by a perfect low pass filter with unity gain and a cutoff frequency of 5 kHz.

a) If there is to be no aliasing in the sampling system, then what is the maximum frequency contained in $x(t)$?

b) If your objective is to use this system to implement a low pass filter with a cut off frequency of 1 kHz, then what cut off frequency should the filter $H(e^{j\omega})$ have?

c) If $H(e^{j\omega})$ is a low pass filter with cut off $0 < \omega_o \leq \pi$, and unity gain in its pass band, then what is $h(n)$ the impulse response of the filter?



(a) $f_s \geq 2f_{max}$ to prevent aliasing

$\therefore f_{max} = \frac{f_s}{2} = \frac{10}{2} = 5$ kHz

(b) $H(e^{j\omega})$ perfect discrete lowpass filter with ω_c that corresponds to 1 kHz in continuous time

$\Omega_c = 2\pi f$

$\omega_c = \frac{\Omega_c}{f_s} = \frac{2\pi(1\text{ kHz})}{10\text{ kHz}} = \frac{\pi}{5}$ radians

(c) $h(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(e^{j\omega}) e^{j\omega n} d\omega$

$= \frac{1}{2\pi}\int_{-\omega_o}^{\omega_o} e^{j\omega n} d\omega = \frac{1}{2\pi}\left(\frac{1}{jn} e^{j\omega n}\right)\Big|_{-\omega_o}^{\omega_o} = \frac{1}{j2\pi n}\left(e^{j\omega_o n} - e^{-j\omega_o n}\right)$

$= \frac{1}{\pi n}\frac{1}{2j}\left(e^{j\omega_o n} - e^{-j\omega_o n}\right) = \frac{1}{\pi n}\sin\omega_o n$

$= \frac{\omega_o}{\pi}\,\text{sinc}\left(\frac{\omega_o}{\pi} n\right)$