EE 438 Midterm Exam #3

November 29, Fall 2007

Name: Solutions

Instructions:

- Follow all instructions carefully!
- This is a 50 minute exam containing **three** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Function types
  - *Right-sided:* $\exists N s.t. \forall n < N, x(n) = 0$
  - *Left-sided:* $\exists N s.t. \forall n > N, x(n) = 0$
  - *Causal:* $\forall n < 0, x(n) = 0$
  - *Anticausal:* $\forall n > 0, x(n) = 0$
  - *Two-sided:* {x(n) is not right-sided} and {x(n) is not left-sided}

- CTFS

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T} \end{aligned}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ x(t) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_o t} \overset{CTFT}{\Leftrightarrow} X(f - f_o)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

$$e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{j2\pi f + a}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{aligned}$$

- DTFT Properties

$$x(-n) \overset{DTFT}{\Leftrightarrow} X(e^{-j\omega})$$

$$x(n - n_o) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})e^{-j\omega n_0}$$

$$x(n)e^{j\omega_o t} \overset{DTFT}{\Leftrightarrow} X(e^{j(\omega-\omega_o)})$$

$$x(n)y(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{2\pi}X(e^{j\omega}) * Y(e^{j\omega})$$

$$x(n) * y(n) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})Y(e^{j\omega})$$

- Z-transform

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ x(n) &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \end{aligned}$$

- DFT

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- DFT

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k)\delta(t - kT) \\ S(f) &= Y\left(e^{j2\pi fT}\right) \end{aligned}$$

- Decimation and Interpolation

$$\begin{aligned} y(n) &= x(nD) \\ Y(e^{j\omega}) &= \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j\frac{\omega - 2\pi k}{D}}\right) \\ s(n) &= \sum_{k=-\infty}^{\infty} y(k)\delta(n - kL) \\ S(e^{j\omega}) &= Y(e^{j\omega L}) \end{aligned}$$

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Problem 1.(34pt) Consider the following transfer functions for LTI systems with right-sided impulse responses,

$$G(z) = (1 - az^{-1})$$
$$H(z) = \frac{1}{(1 - bz^{-1})}$$

where a and b are two real numbers.

a) Specify the ROC for $G(z)$; calculate $g(n)$; and draw a flow diagram that implements this filter.

b) Specify the ROC for $H(z)$; calculate $h(n)$; and draw a flow diagram that implements this filter.

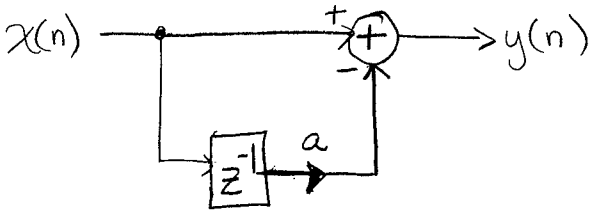
c) Specify the ROC for $F(z) = \frac{G(z)}{H(z)}$; calculate $f(n)$; and draw a flow diagram that implements this filter.

d) For what values of a and b is the system $F(z) = \frac{G(z)}{H(z)}$ stable? Justify your answer.

Note: The impulse responses are right sided.

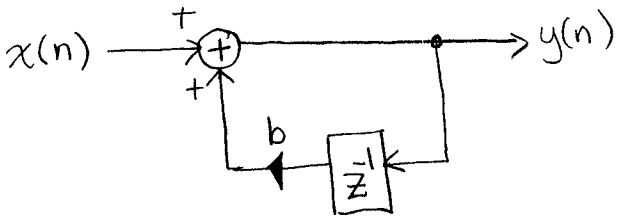
a) $(1 - az^{-1})$ $\Omega_G = \{z \in \mathbb{C} \mid |z| > 0\}$

$g(n) = \delta(n) - a\delta(n-1)$



b) $\frac{1}{(1 - bz^{-1})}$ $\Omega_H = \{z \in \mathbb{C} \mid |z| > |b|\}$

$h(n) = b^n u(n)$

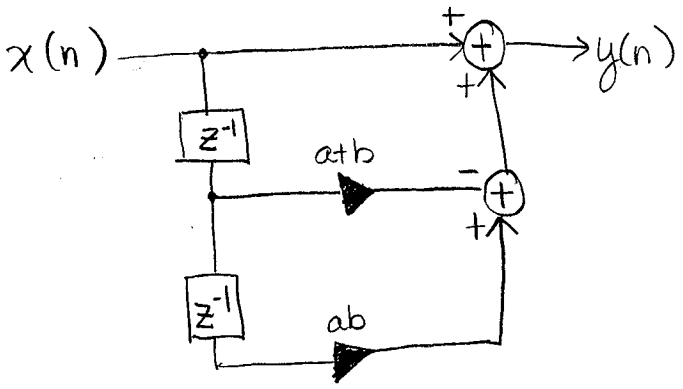


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c) $F(z) = \dfrac{G(z)}{H(z)}$ $\qquad$ **ROC: $\Omega_F = \{z \in \mathbb{C} \mid |z| > 0\}$**

$$F(z) = (1 - az^{-1})(1 - bz^{-1})$$
$$= 1 - (a+b)z^{-1} + abz^{-2}$$

$$\Rightarrow \boxed{f(n) = \delta(n) - (a+b)\delta(n-1) + ab\delta(n-2)}$$



or equivalent

d) $|a| \geq 0$ and $|b| \geq 0$ since the ROC for $F(z)$ is $|z| > 0$ and contains 1 regardless of a and b.

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Problem 2.(33pt)

Consider the stable LTI system $H(z) = \frac{1}{1-az^{-1}}$ with input $x(n)$ and output $y(n)$ where a is any real value.

a) For what values of a is the system right-sided? For what values of a is the system left-sided? Justify your answers.

b) Calculate $y(n)$ when $x(n) = e^{j\omega_0 n}$.

c) Calculate $y(n)$ when $x(n) = \delta(n)$. (Hint: Your answer will depend on the value of a.)

d) Assuming that $a = 2$, specify a flow diagram that implements this system.

(a)

Note that the system is stable, i.e. $|z|=1 \subset \Omega_H$.

Thus, if $1 < |a|$, then the system must be left-sided.

and if $1 > |a|$, then the system must be right-sided.

If $1 = |a|$, then the system is unstable.

(b) $H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$ $\quad X(e^{j\omega}) = 2\pi\delta(\omega-\omega_0)$

$$Y(e^{j\omega}) = 2\pi\delta(\omega-\omega_0)\left(\frac{1}{1-ae^{-j\omega}}\right) = 2\pi\delta(\omega-\omega_0)\left(\frac{1}{1-ae^{-j\omega_0}}\right)$$

$$\boxed{y(n) = \left(\frac{1}{1-ae^{-j\omega_0}}\right)e^{j\omega_0 n}}$$

(c) $x(n) = \delta(n) \Rightarrow X(z) = 1$

$$Y(z) = X(z)H(z) = \frac{1}{1-az^{-1}}$$

$$\boxed{\begin{array}{l}\text{If } |a|>1,\ h(n) = -a^n u(-n-1) \\ \text{If } |a|<1,\ h(n) = a^n u(n)\end{array}}$$

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d) $$H(z) = \frac{1}{1 - az^{-1}}$$

If $a = 2$,

$$H(z) = \frac{1}{1 - 2z^{-1}} \quad \text{and since} \quad H(z) = \frac{Y(z)}{X(z)}$$

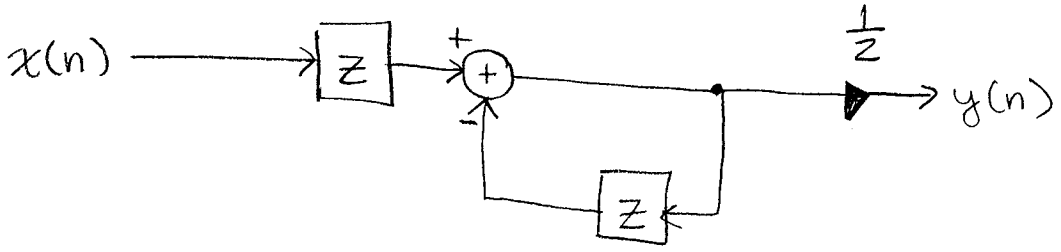
$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 2z^{-1}}$$

$$\Rightarrow Y(z)(1 - 2z^{-1}) = X(z)(1)$$

$$\Rightarrow y(n) - 2y(n-1) = x(n)$$

Because the system must be left-sided when $a = 2$,

$$2y(n-1) = y(n) - x(n)$$
$$y(n-1) = \frac{1}{2}(y(n) - x(n))$$
$$y(n) = \frac{1}{2}(y(n+1) - x(n+1))$$



Solutions

Problem 3.(33pt)

a) Calculate $X_N(k)$, the N-point DFT of $x(n) = \cos\left(\frac{2\pi l}{N} n\right)$ for $0 \le n < N$.

b) Sketch $X_N(k)$ for $0 \le k < N$.

c) Calculate ~~e~~ $W_N(k)$

$$w(n) = \begin{cases} 1 & \text{if } 0 \le n < L \\ 0 & \text{otherwise} \end{cases}$$

where $N \ge L$.

d) Sketch $|W_N(k)|$, for $L = 5$ and $N >> L$.

e) If $w(n)$ is being used to window a signal $x(n)$, what is the advantage of making N larger when L is fixed?

a) $$X_N(k) = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi l}{N} n\right) e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} e^{j\frac{2\pi l}{N}n} e^{-j\frac{2\pi k n}{N}} + \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi l}{N}n} e^{-j\frac{2\pi k n}{N}}$$

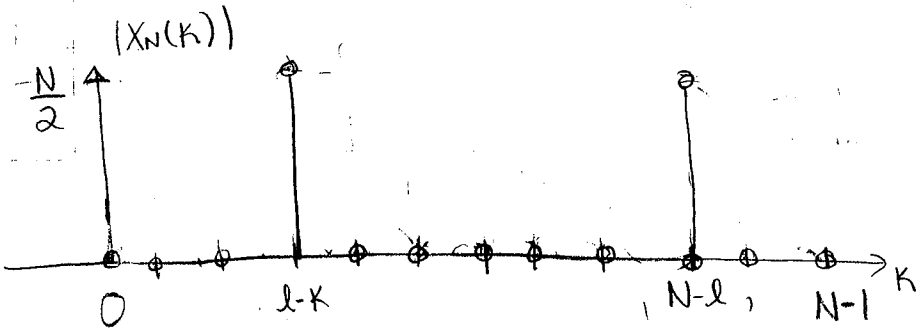
$$= \frac{1}{2}\left(\frac{1 - e^{j2\pi(l-k)}}{1 - e^{j\frac{2\pi}{N}(l-k)}}\right) + \frac{1}{2}\left(\frac{1 - e^{j2\pi(l+k)}}{1 - e^{j\frac{2\pi}{N}(l+k)}}\right)$$

$$X_N(k) = \frac{1}{2}\frac{e^{-j\pi(l-k)}\sin(\pi(l-k))}{e^{-j\frac{\pi}{N}(l-k)}\sin\left(\frac{\pi}{N}(l-k)\right)} + \frac{1}{2}\frac{e^{-j\pi(l+k)}\sin(\pi(l+k))}{e^{-j\frac{\pi}{N}(l+k)}\sin\left(\frac{\pi}{N}(l+k)\right)}$$

Assuming l is an integer s.t $0 \le l < N$

$$X_N(k) = \frac{N}{2}\delta(l-k) + \frac{N}{2}\delta(N-l-k) \quad ; k \in [0, N)$$

b)



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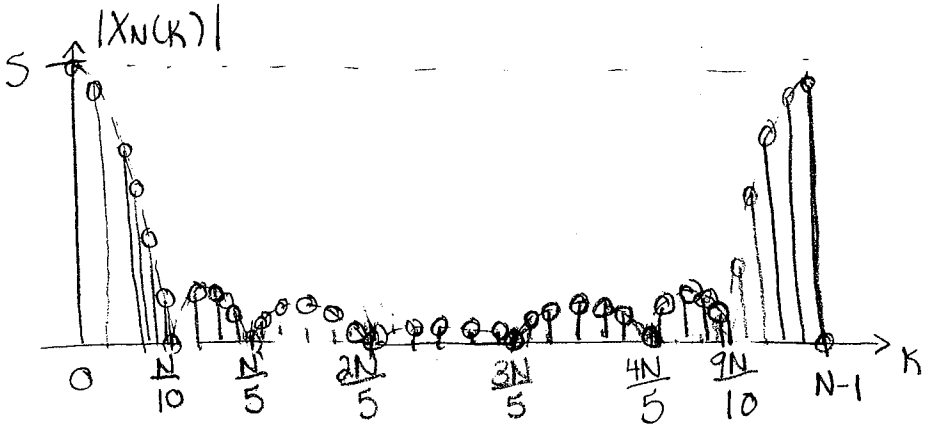
c) $w(n) = \begin{cases} 1 & \text{for } n \in [0, L) \\ 0 & \text{otherwise} \end{cases}$

$$W_N(k) = \sum_{n=0}^{N-1} w(n)\, e^{-j\frac{2\pi kn}{N}}$$

$$W_N(k) = \sum_{n=0}^{L-1} e^{-j\frac{2\pi kn}{N}} = \frac{1 - e^{-j\frac{2\pi kL}{N}}}{1 - e^{-j\frac{2\pi k}{N}}}$$

$$W_N(k) = \frac{e^{-j\frac{\pi kL}{N}} \sin\left(\frac{\pi kL}{N}\right)}{e^{-j\frac{\pi k}{N}} \sin\left(\frac{\pi k}{N}\right)} = \boxed{e^{-j\frac{\pi(L-1)k}{N}} \frac{\sin\left(\frac{\pi L}{N}k\right)}{\sin\left(\frac{\pi k}{N}\right)}}$$

d) $L = 5$ and $N \gg L$



e) If $N \gg L$, then the DFT of the window looks more like the DTFT of the window. And we can get a better idea of what the DTFT of the windowed signal looks like.