EE 438 Midterm Exam #2

October 25, Fall 2007

Name: _____

Instructions:

- Follow all instructions carefully!
- This is a 50 minute exam containing **three** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFS

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

- CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_o t} \overset{CTFT}{\Leftrightarrow} X(f-f_o)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

$$e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{j2\pi f + a}$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- DTFT Properties

$$x(-n) \overset{DTFT}{\Leftrightarrow} X(e^{-j\omega})$$

$$x(n-n_o) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})e^{-j\omega n_0}$$

$$x(n)e^{j\omega_o t} \overset{DTFT}{\Leftrightarrow} X(e^{j(\omega-\omega_o)})$$

$$x(n)y(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{2\pi}X(e^{j\omega}) * Y(e^{j\omega})$$

$$x(n) * y(n) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})Y(e^{j\omega})$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- DFT

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$x(n) = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{2\pi T}\right)$$

$$s(t) = \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT)$$

$$S(f) = Y\left(e^{j2\pi fT}\right)$$

- Decimation and Interpolation

$$y(n) = x(nD)$$

$$Y(e^{j\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j\frac{\omega-2\pi k}{D}}\right)$$

$$s(n) = \sum_{k=-\infty}^{\infty} y(k)\delta(n-kL)$$

$$S(e^{j\omega}) = Y(e^{j\omega L})$$

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Problem 1.(34pt) Consider a discrete-time LTI system with input $x(n)$, output $y(n)$ and impulse response

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(\omega^2+j\omega)} e^{j\omega n} d\omega \, .$$

a) Calculate the DTFT of the function $h(n)$, i.e. $H(e^{j\omega})$ for $|\omega| < \pi$.

b) Is the output of the system real-valued when the input is real-valued? Justify your answer.

c) Compute the $|y(n)|$, when the input is given by $x(n) = e^{j\omega_o n}$.

d) Compute the $y(n)$, when the input is given by $x(n) = \cos(\omega_o n)$.

a) $h(n) = IDTFT\{H(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$

$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(\omega^2+j\omega)} e^{j\omega n} d\omega$

$\Rightarrow \boxed{H(e^{j\omega}) = e^{-(\omega^2+j\omega)} \quad \text{for } |\omega| < \pi}$

b) The output $y(n)$ is real whenever $Y^*(\omega) = Y(-\omega)$.
Let $x(n)$ denote the input and $X(\omega)$ denote $DTFT\{x(n)\}$.

$Y(\omega) = X(\omega)H(\omega)$

$Y^*(\omega) = (X(\omega)H(\omega))^* = X^*(\omega)H^*(\omega)$

$Y(-\omega) = (X(-\omega)H(-\omega)) = X(-\omega)H(-\omega)$

if $x(n)$ is real, then $X^*(\omega) = X(-\omega)$. Thus,
$H^*(\omega) = H(-\omega)$ is the condition for $Y^*(\omega) = Y(-\omega)$,
i.e. $y(n)$ real.

$H^*(\omega) = e^{-(\omega^2 - j\omega)} = e^{(j\omega - \omega^2)}$

$H(-\omega) = e^{-((-\omega)^2 + j(-\omega))} = e^{(j\omega - \omega^2)}$

Thus $H^*(\omega) = H(-\omega)$ and $y(n)$ is real when $x(n)$ is real.

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c) $|y(n)| = ?$

$X(e^{j\omega}) = 2\pi\delta(\omega-\omega_0) \;;\; |\omega| < \pi$

$Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$

$= 2\pi\delta(\omega-\omega_0)\, e^{-(\omega^2+j\omega)} \;;\; |\omega| < \pi$

$= \left(e^{-(\omega_0^2+j\omega_0)}\right)\left(2\pi\delta(\omega-\omega_0)\right)$

$y(n) = e^{-(\omega_0^2+j\omega_0)}\, e^{j\omega_0 n}$

$= e^{-\omega_0^2}\, e^{j(\omega_0 n-\omega_0)} \Rightarrow \boxed{|y(n)| = e^{-\omega_0^2}}$

d) $X(e^{j\omega}) = \pi\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right) \;;\; |\omega| < \pi$

$Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$

$= \pi\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right) e^{-(\omega^2+j\omega)} \;;\; |\omega| < \pi$

$= \left(e^{-(\omega_0^2+j\omega_0)}\right)\left(\pi\delta(\omega-\omega_0)\right) + \left(e^{-((-\omega_0)^2+j(-\omega_0))}\right)\left(\pi\delta(\omega+\omega_0)\right)$

$= e^{-(\omega_0)^2}\left(\pi\delta(\omega-\omega_0)\, e^{-j\omega_0} + \pi\delta(\omega+\omega_0)\, e^{j\omega_0}\right)$

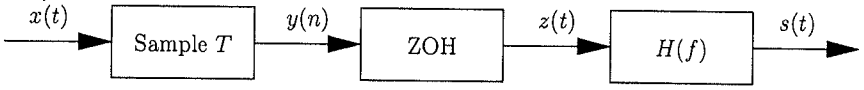
$y(n) = e^{-(\omega_0)^2}\left(\frac{1}{2} e^{j\omega_0 n} e^{-j\omega_0} + \frac{1}{2} e^{-j\omega_0 n} e^{j\omega_0}\right)$

$= e^{-(\omega_0)^2}\left(\frac{1}{2}\left(e^{j(\omega_0 n-\omega_0)} + e^{-j(\omega_0 n-\omega_0)}\right)\right)$

$\boxed{y(n) = e^{-(\omega_0)^2}\cos\left(\omega_0(n-1)\right)} = |H(e^{j\omega_0})|\cos(\omega_0 n + \angle H(e^{j\omega_0}))$

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Problem 2.(33pt)



Consider the sampling system shown above.

a) Calculate expressions for the signals $Y(e^{j\omega})$, $Z(f)$, and $S(f)$ in terms of $X(f)$ and $H(f)$.

For parts b) and c) below, assume that $x(t)$ is band limited to $\frac{1}{2T}$.

b) Calculate an expression for $S(f)$ when $H(f)$ is a low pass filter with a cut off frequency of $\frac{1}{2T}$ and a gain of 1 in the pass band.

c) Specify a frequency response $H(f)$ so that $x(t) = s(t)$, and sketch the function $H(f)$.

a) $\boxed{Y(e^{j\omega}) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X\left(\frac{\omega - 2\pi K}{2\pi T}\right)}$

$$Z(f) = Y(e^{j(2\pi T f)})\, T\,\mathrm{sinc}(Tf)$$

$$= \left(\frac{1}{T}\sum_{K=-\infty}^{\infty} X\left(\frac{2\pi T f - 2\pi K}{2\pi T}\right)\right)\left(T\,\mathrm{sinc}(Tf)\right)$$

$$\boxed{Z(f) = \sum_{K=-\infty}^{\infty} X\left(f - \frac{K}{T}\right)\mathrm{sinc}(Tf)}$$

$$S(f) = Z(f)\,H(f)$$

$$\boxed{S(f) = \sum_{K=-\infty}^{\infty} X\left(f - \frac{K}{T}\right)\mathrm{sinc}(Tf)\,H(f)}$$

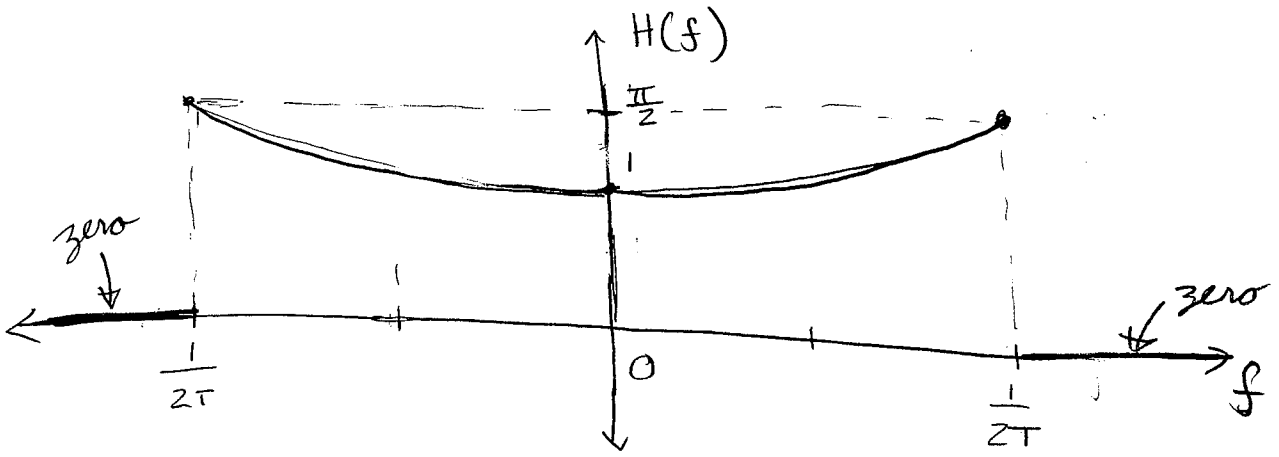
Note: In this solution ZOH uses the unshifted rectangle function. However, the shifted version is also correct.

b) $H(f) = \mathrm{rect}\left(\frac{f}{T}\right) \Rightarrow \boxed{S(f) = X(f)\,\mathrm{sinc}(Tf)}$

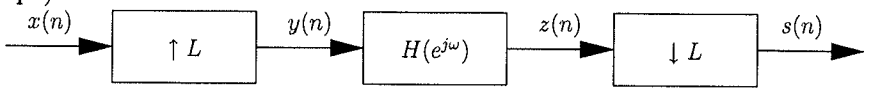
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c) $$H(f) = \frac{\text{rect}\left(\frac{f}{T}\right)}{\text{sinc}(Tf)}$$

$$= \frac{\pi T f}{\sin(\pi T f)} \text{rect}\left(\frac{f}{T}\right)$$



Problem 3.(33pt)



Consider the discrete-time system shown above with a general filter $H(e^{j\omega})$.

a) Calculate expressions for the signals $Y(e^{j\omega})$, $Z(e^{j\omega})$, and $S(e^{j\omega})$ in terms of $X(e^{j\omega})$ and $H(e^{j\omega})$.

b) Calculate an expression for the function $\frac{S(e^{j\omega})}{X(e^{j\omega})}$.

For parts c) and d) assume that $H(e^{j\omega})$ is a perfect LPF with a cut-off of π/L, and a pass-band gain of L.

c) Calculate the impulse response $h(n)$.

d) Calculate an expression for the function $\frac{S(e^{j\omega})}{X(e^{j\omega})}$.

a) $\boxed{Y(e^{j\omega}) = X(e^{jL\omega})}$

$Z(e^{j\omega}) = Y(e^{j\omega})\,H(e^{j\omega})$

$\boxed{Z(e^{j\omega}) = X(e^{jL\omega})\,H(e^{j\omega})}$

$$S(e^{j\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} Z\left(e^{j\left(\frac{\omega-2\pi k}{L}\right)}\right)$$

$$= \frac{1}{L}\sum_{k=0}^{L-1} X\left(e^{jL\left(\frac{\omega-2\pi k}{L}\right)}\right) H\left(e^{j\left(\frac{\omega-2\pi k}{L}\right)}\right)$$

$$= \frac{1}{L}\sum_{k=0}^{L-1} X\left(e^{j(\omega-2\pi k)}\right) H\left(e^{j\left(\frac{\omega-2\pi k}{L}\right)}\right)$$

$$\boxed{S(e^{j\omega}) = \frac{1}{L}X(e^{j\omega})\sum_{k=0}^{L-1} H\left(e^{j\left(\frac{\omega-2\pi k}{L}\right)}\right)}$$

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b) $$\frac{S(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{1}{L}X(e^{j\omega})\sum_{k=0}^{L-1} H\left(e^{j\left(\frac{\omega-2\pi k}{L}\right)}\right)}{X(e^{j\omega})}$$

$$\boxed{\frac{S(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{L}\sum_{k=0}^{L-1} H\left(e^{j\left(\frac{\omega-2\pi k}{L}\right)}\right)}$$

For parts c and d, $H(e^{j\omega}) = L\,\text{rect}\left(\frac{L\omega}{2\pi}\right)$

c) $$h(n) = L\,\frac{\sin\frac{\pi}{L}n}{\pi n} = \frac{\sin\frac{\pi}{L}n}{\frac{\pi n}{L}} = \text{sinc}\left(\frac{n}{L}\right)$$

$$\boxed{h(n) = \text{sinc}\left(\frac{n}{L}\right)}$$

d) $$\frac{S(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{L}\sum_{k=0}^{L-1} H\left(e^{j\left(\frac{\omega-2\pi k}{L}\right)}\right)$$

$$= \frac{1}{L}\sum_{k=0}^{L-1} L\,\text{rect}\left(\frac{L\left(\frac{\omega-2\pi k}{L}\right)}{2\pi}\right)$$

$$= \sum_{k=0}^{L-1} \text{rect}\left(\frac{\omega-2\pi k}{2\pi}\right)$$

$$\boxed{\frac{S(e^{j\omega})}{X(e^{j\omega})} = 1 \quad |\omega| < \pi}$$