EE 438 Midterm Exam #1

September 28, Fall 2007

Name: Solutions

Instructions:

- Follow all instructions carefully!
- This is a 50 minute exam containing **four** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFS

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

- CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi t}dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi t}d\omega$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_o t} \overset{CTFT}{\Leftrightarrow} X(f-f_o)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

$$e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{j2\pi f + a}$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- DTFT Properties

$$x(-n) \overset{DTFT}{\Leftrightarrow} X(e^{-j\omega})$$

$$x(n-n_o) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})e^{-j\omega n_0}$$

$$x(n)e^{j\omega_o t} \overset{DTFT}{\Leftrightarrow} X(e^{j(\omega-\omega_o)})$$

$$x(n)y(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{2\pi}X(e^{j\omega}) * Y(e^{j\omega})$$

$$x(n) * y(n) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})Y(e^{j\omega})$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- DFT

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$x(n) = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right)$$

$$s(t) = \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT)$$

$$S(f) = Y\left(e^{j2\pi fT}\right)$$

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Problem 1.(34pt) Let $T[\cdot]$ be a Discrete-time Linear time invariant system with input $x(n)$ and output $y(n)$; so that $y(n) = T[x(n)]$.

a) Prove that $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ where $h(n) = T[\delta(n)]$.

b) Use the result of a) above to show that when $x(n) = e^{j\omega_o n}$, then $y(n) = Ce^{j\omega_o n}$, and calculate an expression for the complex number C in terms of $h(n)$.

c) For this part, assume that $h(n)$ is real valued. Then calculate the output $y(n)$ when $x(n) = \cos(\omega_o n)$ in terms of the complex number $C = Ae^{j\phi}$ from part c) above. (Do not use the Fourier transform to do this.)

a) $y(n) = T[x(n)]$

$h(n) = T[\delta(n)]$

$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \Rightarrow$

$\Rightarrow y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$ (*)

Since T is linear we can treat $x(k)$ like scalar valued multiples of $\delta(n-k)$ and rewrite (*) as follows.

$y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)]$ (**)

Since T is TI and $T[\delta(n)] = h(n)$,

$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ from (**). Q.E.D.

b) $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} e^{j\omega_o k}h(n-k)$

Let $l = n-k$, then $y(n) = \sum_{l=-\infty}^{\infty} e^{j\omega_o(n-l)}h(l)$

$\Rightarrow k = n-l$

$= e^{j\omega_o n}\sum_{l=-\infty}^{\infty} e^{-j\omega_o l}h(l)$

$= e^{j\omega_o n}H(\omega_o)$

$\Rightarrow \boxed{C = H(\omega_o)}$

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c) $h[n]$ real $\Leftrightarrow H(\omega) = H(-\omega)^*$

Since $\cos\omega_o n = \frac{1}{2}\left(e^{-j\omega_o n} + e^{j\omega_o n}\right) = x(n)$

$T[\,]$ is LTI and $y(n) = H(\omega_o)e^{j\omega_o n}$

whenever $x(n) = e^{j\omega_o n}$,

$$y(n) = \frac{1}{2}\left(H(-\omega_o)e^{-j\omega_o n} + H(\omega_o)e^{j\omega_o n}\right)$$

whenever $x(n) = \cos\omega_o n$.

Because $H(\omega_o) = H(-\omega_o)^*$ since $h[n]$ is rea

$H(-\omega_o) = H(\omega_o)^*$.

$$\Rightarrow y(n) = \frac{1}{2}\left(H(\omega_o)^* e^{-j\omega_o n} + H(\omega_o)e^{j\omega_o n}\right)$$

$$= \frac{1}{2}\left(|H(\omega_o)|e^{-j\angle H(\omega_o)}e^{-j\omega_o n} + |H(\omega_o)|e^{j\angle H(\omega_o)}e^{j\omega_o n}\right)$$

$$\boxed{y(n) = \frac{1}{2}|H(\omega_o)|\cos(\omega_o n + \angle H(\omega_o))}$$

Problem 2.(33pt) Consider a continuous-time signal

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} e^{j2\pi kt}$$

which is the input to a CT LTI system with transfer function $H(f)$ and output $y(t)$.

a) Calculate the CTFT of $x(t)$.

b) Is $x(t)$ a real valued or complex function? Is it an even function, odd function, or has no symmetry? Justify your answer.

c) Give an expression for $y(t)$ in terms of $H(f)$.

d) Give conditions which guarantee that the output of the system will be real valued.

a) Since $x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} e^{j2\pi kt} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt}$

where a_k are the coefficients of the Fourier Series.

$$\Rightarrow \boxed{X(f) = \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} \delta(f-k)}$$

b) $X(f)$ is real and even $\Rightarrow$ $\boxed{x(t) \text{ is real and even}}$

$X(f)$ is even since $X(f) = X(-f)$

$$X(-f) = \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} \delta(-f-k) = \sum_{-l=-\infty}^{+\infty} \frac{1}{(-l)^2+1} \delta(-f+l)$$

$$= \sum_{l=-\infty}^{\infty} \frac{1}{l^2+1} \delta(-(f-l))$$

$$= \sum_{l=-\infty}^{\infty} \frac{1}{l^2+1} \delta(f-l) = X(f)$$

Since $\frac{1}{k^2+1}$ is real $\forall k \in \mathbb{N}$, $X(f)$ is real

c) $Y(f) = H(f)X(f)$

$$= H(f) \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} \delta(f-k)$$

$$= \sum_{k=-\infty}^{\infty} \frac{H(k)}{k^2+1} \delta(f-k) \Rightarrow \boxed{y(t) = \sum_{k=-\infty}^{\infty} \frac{H(k)}{k^2+1} e^{j2\pi kt}}$$

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d) $y(t)$ is real if $Y(f) = Y(-f)^*$

$$\Rightarrow \frac{H(K)}{K^2+1} = \left(\frac{H(-K)}{(-K)^2+1}\right)^* \quad \text{must be true}$$

$$\Rightarrow \frac{H(K)}{K^2+1} = \frac{H(-K)^*}{K^2+1}$$

$$\Rightarrow H(K) = H(-K)^*$$

or $\boxed{H(f) = H(-f)^*}$ for $y(t)$ to be real.

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Problem 3.(33pt) Consider the following recursive difference equation

$$y(n) = x(n) + ay(n-1)$$

with input $x(n)$ and output $y(n)$.

a) Show that the system represented by the difference equation is a linear system.

b) Show that the system represented by the difference equation is a time-invariant system.

c) Use the DTFT to calculate the frequency response $H(e^{j\omega})$.

d) Use the result of a) to calculate the impulse response $h(n)$.

a) Let $y_1(n) = T[x_1[n]] = x_1(n) + a y_1(n-1)$
and $y_2(n) = T[x_2[n]] = x_2(n) + a y_2(n-1)$
Then suppose $x(n) = \alpha x_1(n) + \beta x_2(n)$
$y(n) = \alpha y_1(n) + \beta y_2(n)$

$\Rightarrow y(n) = \alpha y_1(n) + \beta y_2(n) = \alpha(x_1(n) + a y_1(n-1)) + \beta(x_2(n) + a y_2(n-$

$= \alpha x_1(n) + \beta x_2(n) + \alpha y_1(n-1) + \beta y_2(n-1)$

$= x(n) + y(n-1)$

$\Rightarrow y(n) = T[x(n)]$

Thus, $y(n) = \alpha y_1(n) + \beta y_2(n) = T[\alpha x_1(n) + \beta x_2(n)]$
by definition of $T[\,]$. Q.E.D

b) Let $x_2(n) = x_1(n - n_0)$ and $y_1(n) = x_1(n) + a y_1(n-1)$
and $y_2(n) = x_2(n) + a y_2(n-1)$

$y_1(n - n_0) = x_1(n - n_0) + a y_1(n - n_0 - 1)$
$\Rightarrow x_1(n - n_0) = y_1(n - n_0) - a y_1(n - n_0 - 1)$

$y_2(n) = x_2(n) + a y_2(n-1)$
$y_2(n) - a y_2(n-1) = x_2(n) = x_1(n - n_0) = y_1(n - n_0) - a y(n - n_0 - 1)$

$\Rightarrow y_2(n) = y_1(n - n_0)$. Thus, the system is TI by definition.

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c) $y(n) = x(n) + a\,y(n-1)$

$Y(\omega) = X(\omega) + a\,Y(\omega)e^{-j\omega}$

$Y(\omega)(1 - ae^{-j\omega}) = X(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \boxed{\frac{1}{1-ae^{-j\omega}}}$$

d) $$a^n u(n) \overset{DTFT}{\Longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

Thus, $h(n) = IDTFT\left\{\frac{1}{1-ae^{-j\omega}}\right\}$

$$\Rightarrow \boxed{h(n) = a^n u(n)}$$