EE 438 Final Exam

December 12, Fall 2007

Name: _____

Instructions:

- Follow all instructions carefully!
- This is a 110 minute exam containing **six** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Function types
  - *Right-sided:* $\exists N s.t. \forall n < N, x(n) = 0$
  - *Left-sided:* $\exists N s.t. \forall n > N, x(n) = 0$
  - *Causal:* $\forall n < 0, x(n) = 0$
  - *Anticausal:* $\forall n > 0, x(n) = 0$
  - *Two-sided:* {x(n) is not right-sided} and {x(n) is not left-sided}

- CTFS

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T} \end{aligned}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ x(t) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_o t} \overset{CTFT}{\Leftrightarrow} X(f-f_o)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

$$e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{j2\pi f + a}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{aligned}$$

- DTFT Properties

$$x(-n) \overset{DTFT}{\Leftrightarrow} X(e^{-j\omega})$$

$$x(n-n_o) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})e^{-j\omega n_0}$$

$$x(n)e^{j\omega_o t} \overset{DTFT}{\Leftrightarrow} X(e^{j(\omega-\omega_o)})$$

$$x(n)y(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{2\pi}X(e^{j\omega}) * Y(e^{j\omega})$$

$$x(n) * y(n) \overset{DTFT}{\Leftrightarrow} X(e^{j\omega})Y(e^{j\omega})$$

- Z-transform

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ x(n) &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \end{aligned}$$

- DFT

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT) \\ S(f) &= Y\left(e^{j2\pi fT}\right) \end{aligned}$$

- Decimation and Interpolation

$$
\begin{aligned}
y(n) &= x(nD) \\
Y(e^{j\omega}) &= \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j\frac{\omega-2\pi k}{D}}\right) \\
s(n) &= \sum_{k=-\infty}^{\infty} y(k)\delta(n-kL) \\
S(e^{j\omega}) &= Y(e^{j\omega L})
\end{aligned}
$$

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Problem 1.(20pt) For each of the following discrete-time systems with input $x(n)$ and output $y(n)$, state whether the system is i) linear or non-linear, ii) time-invariant or time varying, iii) memoryless or memory, and iv) BIBO stable or unstable. (No derivation or proof is required for this problem.)

a) $y(n) = x(n)$

b) $y(n) = \frac{1}{x(n)}$

c) $y(n) = \sum_{k=-\infty}^{n} x(k)$

d) $y(n) = x(n)e^{-n}$

e) $y(n) = e^{x(n)}$.

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Problem 2.(10pt)

Let $y(n) = T[x(n)]$ be a discrete time LTI system with input $x(n)$ and output $y(n)$, and let $h(n) = T[\delta(n)]$ be the response to an impulse at time zero. Derive an expression for the output $y(n)$ for any input $x(n)$.

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Problem 3.(10pt)
Let $y(n) = x(n) * h(n)$ be a discrete-time LTI system with input $x(n)$, output $y(n)$, and real-valued impulse response $h(n)$.

a) Prove that when the input has the form $x(n) = e^{j\omega_o n}$ the output has the form $y(n) = Ce^{j\omega_o n}$ and derive an expression for C.

b) Use this result to calculate $y(n)$ when $x(n) = \cos(\omega_o n)$ where $C = Ae^{j\theta}$.

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Problem 4.(20pt)
Consider the discrete-time LTI system described by the difference equation

$$y(n) = ay(n-1) + x(n)$$

where a is a real value, $x(n)$ is the input and $y(n)$ is the output.

a) Find the transfer function $H(z)$ of the system.

b) Compute the right sided impulse response of the system and specify its ROC.

c) Compute the left sided impulse response of the system and specify its ROC.

d) In this part, we will assume that the system is known to be stable. Give the impulse response for $a = 2$.

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Problem 5.(20pt)
Let $x(n) = u(n) - u(n-L)$.

a) Calculate $X(z)$, the z-transform of $x(n)$, and specify the ROC.

b) Calculate $X(e^{j\omega})$, the DTFT of $x(n)$.

c) Sketch $|X(e^{j\omega})|$ for $L = 5$ on the interval $[0, 2\pi]$.

d) Calculate $X_N(k)$ the DFT of $x(n)$ for $N \geq L$.

e) Sketch $|X_N(k)|$ the magnitude of the DFT of $x(n)$ for $N = 10$ and $L = 5$.

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Problem 6.(20pt)
A CT signal $x(t)$ is sampled at 10 kHz to form a DT signal $y(n)$. The signal $y(n)$ is then filtered with the DT filter $H(e^{j\omega})$ to form a signal $r(n)$. Finally, the signal $r(n)$ is reconstructed to form $s(t)$ by using a ZOH followed by a perfect low pass filter with unity gain and a cutoff frequency of 5 kHz.

a) If there is to be no aliasing in the sampling system, then what is the maximum frequency contained in $x(t)$?

b) If your objective is to use this system to implement a low pass filter with a cut off frequency of 1 kHz, then what cut off frequency should the filter $H(e^{j\omega})$ have?

c) If $H(e^{j\omega})$ is a low pass filter with cut off $0 < \omega_o \leq \pi$, and unity gain in its pass band, then what is $h(n)$ the impulse response of the filter?

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