EE 301 Midterm Exam #2

October 23, Fall 2009

**Name:** _____

**Instructions:**

- Follow all instructions carefully!
- This is a 50 minute exam containing **four** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

# Good Luck.

# Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Continuous Time Fourier Series (CTFS)

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\frac{2\pi}{T}t}dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \end{aligned}$$

- CTFS Properties

$$x(t-t_o) \overset{CTFS}{\Leftrightarrow} a_k e^{-jk\frac{2\pi}{T}t_o}$$

For $x(t)$ real valued $a_k = a_{-k}^*$

$$\frac{dx(t)}{dt} \overset{CTFS}{\Leftrightarrow} jk\frac{2\pi}{T}a_k$$

$$x(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \overset{CTFS}{\Leftrightarrow} Ta_k b_k$$

$$x(t)y(t) \overset{CTFS}{\Leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- CTFT

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega-\omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega)Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(\omega/(2\pi))$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega+a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\sum_{k-\infty}^{\infty} 2\pi\delta(\omega-2\pi k/T)$$

- DFT

$$\begin{aligned} X_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ x(n) &= \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- DTFT

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT) \\ S(\omega) &= Y(\omega T) \end{aligned}$$

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**Problem 1.** (25pt) *Sine wave inputs to LTI systems*

Let $y(t) = S[x(t)]$ be an LTI system with input $x(t)$, output $y(t)$, and real valued impulse response $h(t)$. Determine the output, $y(t)$, in the following cases.

**a)** $x(t) = e^{j\omega_o t}$.

**b)** $x(t) = e^{-j\omega_o t}$.

**c)** $x(t) = \cos(\omega_o t)$.

**d)** $x(t) = B\cos(\omega_o t + \phi)$, where $B$ and $\phi$ are real values.

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**Problem 2.** (25pt) *The continuous time Fourier series*

For each example, $x(t)$ is a period signal with period $T$ and associated frequency $\omega_o = \frac{2\pi}{T}$.

**a)** Calculate the Fourier series coefficients, $a_k$, for the signal, $x(t) = e^{j2\omega_o t} + e^{j5\omega_o t}$.

**b)** Calculate the Fourier series coefficients, $a_k$, for the signal, $x(t) = \cos(2\omega_o t)$.

**c)** Calculate the Fourier series coefficients, $a_k$, for the signal, $x(t) = B\cos(2\omega_o t + \phi)$.

**d)** Calculate a closed form expression for the signal, $x(t)$, when $x(t) = \sum_{k=-\infty}^{\infty} e^{jk\omega_o t}$.

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**Problem 3.**(25pt) *Orthonormal Basis Functions*

Let $\phi_k(t) = e^{j2\pi kt}$ for $k$ an integer be functions on the interval $[0, 1]$ with associated inner product given by

$$\langle \phi_k, \phi_l \rangle = \int_0^1 \phi_k(t)\phi_l^*(t)dt \ .$$

**a)** Prove that the functions $\{\phi_k\}_{k=-\infty}^{\infty}$ are normal.
**b)** Prove that the functions $\{\phi_k\}_{k=-\infty}^{\infty}$ are orthogonal.
**c)** Assume that $x(t)$ is a real valued signal that has the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, \phi_k(t) \ .$$

Then show that Parseval's theorem applies, i.e. that

$$\int_0^1 |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

**d)** Let $\tilde{x}(t)$ be formed by filtering the signal $x(t)$ so that only the first $M$ harmonics are retained. Then the RMS error between these two signals is defined as

$$\text{RMS} = \sqrt{\int_0^1 |x(t) - \tilde{x}(t)|^2 \, dt}$$

Then calculate an expression for the RMS error between $x(t)$ and $\tilde{x}(t)$ in terms of the coefficients $a_k$.

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**Problem 4.**(25pt) *Periodic inputs to LTI systems*

Let $x(t)$ be a period signal with period $T$ and associated frequency $\omega_o = \frac{2\pi}{T}$; and let $a_k$ be the associated Fourier series expansion coefficients for $x(t)$.

The signal $x(t)$ is used as the input to an LTI system with output $y(t)$ and real valued impulse response h(t).

**a)** Show that $y(t)$ is periodic with period $T$.

**b)** Derive an expression for, $b_k$, the Fourier series coefficients of $y(t)$ in terms of the $a_k$'s and coefficients computed from the impulse response, $h(t)$.

**c)** Calculate an expression for $b_k$ when $h(t) = \text{rect}\,(t/T)$.

**d)** Use the result of c) above to compute an explicit expression for $y(t)$.

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